

On radicals of module coalgebras

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Abstract

We introduce the notion of an idempotent radical class of module coalgebras over a bialgebra B . We prove that if \mathcal{R} is an idempotent radical class of B -module coalgebras, then every B -module coalgebra contains a unique maximal B -submodule coalgebra in \mathcal{R} . Moreover, a B -module coalgebra C is a member of \mathcal{R} if, and only if, DB is in \mathcal{R} for every simple subcoalgebra D of C . The collection of B -cocleft coalgebras and the collection of H -projective module coalgebras over a Hopf algebra H are idempotent radical classes. As applications, we use these idempotent radical classes to give another proofs for a projectivity theorem and a normal basis theorem of Schneider without assuming a bijective antipode.

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1. Introduction

In [5], it has been proved that for any module coalgebra C over a finite-dimensional Hopf algebra H , there exists a unique maximal H -submodule coalgebra $\mathcal{P}(C)$ of C such that $\mathcal{P}(C)$ is a projective H -module. More importantly, the (H) -projectivity of C can be completely determined by the projectivity of the H -submodules generated by the simple subcoalgebras of C . Projectivity of modules over any ring, in general, is not preserved under the direct limit. However, projectivity of H -module coalgebras behaves quite differently. Indeed, the collection of projective H -module coalgebras carries certain properties which allow the existence of a unique maximal projective H -submodule subcoalgebra of any given H -module coalgebra. This example is a prototype of idempotent radical classes introduced in the sequel.

In this paper, we will give the definitions of *radical class* and *idempotent radical class* of the category **Coalg- B** of right module coalgebras over a bialgebra B . They are indeed generalizations of the coradical of coalgebras. We show that if \mathcal{R} is an idempotent radical class of **Coalg- B** , then for any $C \in \mathbf{Coalg-}B$, there exists a unique maximal

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subcoalgebra $\mathcal{R}(C)$ of C such that $\mathcal{R}(C)$ is a member of \mathcal{R} . Moreover, $C \in \mathcal{R}$ if, and only if, $DB \in \mathcal{R}$ for all simple subcoalgebras D of C .

To demonstrate the abundance of idempotent radical classes, we shall prove that the following well-known classes of right B -module coalgebras are idempotent radical classes:

- (i) the collection \mathcal{P} of all projective H -module coalgebras over a Hopf algebra H ;
- (ii) the collection \mathcal{E} of all B -module coalgebras C for which the functor

$$\bar{?} : \mathcal{M}_B^C \longrightarrow \mathcal{M}^{\bar{C}}, M \mapsto \bar{M} = M/MB^+$$

defines an equivalence;

- (iii) the collection \mathcal{C} of all cocleft B -module coalgebras.

As applications of these idempotent radical classes, we shall give another proofs for a projectivity theorem and a normal basis theorem of Schneider [7]. Obviously, Radford's freeness theorem [6] for pointed Hopf algebras is an immediate consequence of Schneider's results.

Throughout this paper, we will assume that all the algebras and coalgebras are over the same ground field k unless stated otherwise. We will write $G(C)$ for the set of all group-like elements in a coalgebra C . The tensor product \otimes_k over the base field k will simply be denoted by \otimes , and we always use B to denote a bialgebra over k .

2. Radical classes of module coalgebras

Let B be a bialgebra over a field k . A coalgebra C over k is said to be a (right) B -module coalgebra, or a *module coalgebra over B* if C is a right B -module and the coalgebra structure maps $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow k$ are B -module maps, where $C \otimes C$ is the right B -module with diagonal B -action and k is considered as a trivial B -module. The category of all (right) B -module coalgebras will be denoted by **Coalg- B** .

For any B -module coalgebra C , we will simply call a subcoalgebra of C which is invariant under the B -action a *B -submodule coalgebra* of C .

If X, Y are subspaces of a B -module coalgebra C , recall [8] that the “wedge” $X \wedge Y$ is defined as

$$X \wedge Y = \Delta^{-1}(X \otimes C + C \otimes Y).$$

Following [4,8], we define $\bigwedge^1 X = X$ and $\bigwedge^{n+1} X = X \wedge (\bigwedge^n X)$ for $n \geq 1$. If X contains the coradical C_0 of C , then we have

$$\sum_{n \geq 1} \bigwedge^n X = C.$$

In particular, if both X and Y are B -submodule coalgebras of C , then so is $X \wedge Y$.

For any collection \mathcal{R} of B -module coalgebras, we simply call a member of \mathcal{R} an \mathcal{R} -coalgebra. For $C \in \mathbf{Coalg-}B$, a B -submodule coalgebra of C which is also a member of \mathcal{R} is called an \mathcal{R} -subcoalgebra of C . Now, we can introduce our definition of a radical class.

Definition 2.1. Let B be a bialgebra. A non-empty collection \mathcal{R} of objects in **Coalg- B** is said to be a *radical class* if it satisfies the following conditions:

- (R1) \mathcal{R} is closed under subobjects, i.e. every B -submodule coalgebra of an \mathcal{R} -coalgebra is an \mathcal{R} -coalgebra.
- (R2) For any $C \in \mathbf{Coalg-}B$ and $\{C_i\}_{i \in I}$ a family of \mathcal{R} -subcoalgebras of C , $\sum_{i \in I} C_i \in \mathcal{R}$.

A radical class \mathcal{R} of **Coalg- B** is called *idempotent* if it satisfies:

- (R3) For any $C \in \mathbf{Coalg-}B$, if C_1, C_2 are \mathcal{R} -subcoalgebras of C , then $C_1 \wedge C_2 \in \mathcal{R}$.

Definition 2.2. Let \mathcal{R} be a radical class of **Coalg- B** . For $C \in \mathbf{Coalg-}B$, the \mathcal{R} -radical of C is defined as the sum of all the \mathcal{R} -subcoalgebras of C , and is denoted by $\mathcal{R}(C)$.

It follows immediately from the definition that $\mathcal{R}(C)$ is the unique maximal \mathcal{R} -subcoalgebra of the B -module coalgebra C . Indeed, the notion of the \mathcal{R} -radical of module coalgebras is a generalization of that of the coradical of coalgebras.

Example 2.3. Let B be the trivial bialgebra k . Then $\mathbf{Coalg}\text{-}B$ is simply the category of all coalgebras over k . Consider the collection \mathcal{R} of all semisimple coalgebras over k . By [2, Theorem 3], every subcoalgebra of a semisimple coalgebra is semisimple, and the sum of a collection of semisimple subcoalgebras of a coalgebra C is also semisimple. Therefore, \mathcal{R} is a radical class of $\mathbf{Coalg}\text{-}B$, and $\mathcal{R}(C)$ is the coradical C_0 of C . Obviously, \mathcal{R} is not idempotent.

Like the properties of coradical of coalgebras, the following result is obtained:

Proposition 2.4. Let \mathcal{R} be a radical class of $\mathbf{Coalg}\text{-}B$, and $C \in \mathbf{Coalg}\text{-}B$.

- (i) If D is a B -submodule coalgebra of C , then $\mathcal{R}(D) = \mathcal{R}(C) \cap D$.
- (ii) If C is a direct sum of a family $\{C_i\}$ of B -submodule coalgebras, then $\mathcal{R}(C) = \bigoplus_i \mathcal{R}(C_i)$.

Proof. (i) Clearly, $\mathcal{R}(D) \subseteq \mathcal{R}(C)$ and hence $\mathcal{R}(D) \subseteq \mathcal{R}(C) \cap D$. Conversely, since $\mathcal{R}(C) \cap D$ is an \mathcal{R} -subcoalgebra of D , $\mathcal{R}(C) \cap D \subseteq \mathcal{R}(D)$.

(ii) If $C = \bigoplus_i C_i$ as B -module coalgebras, it follows from [2, Theorem 3] that $\mathcal{R}(C) = \bigoplus_i (\mathcal{R}(C) \cap C_i)$. Hence, by (i), $\mathcal{R}(C) = \bigoplus_i \mathcal{R}(C_i)$. \square

If \mathcal{R} is an idempotent radical class of $\mathbf{Coalg}\text{-}B$, it is immediately seen that

$$\mathcal{R}(C) \wedge \mathcal{R}(C) = \mathcal{R}(C).$$

Moreover, the membership of a B -module coalgebra in \mathcal{R} can be determined by its simple subcoalgebras.

Proposition 2.5. Let \mathcal{R} be an idempotent radical class of $\mathbf{Coalg}\text{-}B$. The following statements concerning a B -module coalgebra C are equivalent:

- (i) C is an \mathcal{R} -coalgebra.
- (ii) There exists an \mathcal{R} -subcoalgebra C' of C which contains the coradical C_0 of C .
- (iii) For every simple subcoalgebra D of C , DB is an \mathcal{R} -subcoalgebra of C .

Proof. (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) follow directly from (R1).

(iii) \Rightarrow (i): Since DB is an \mathcal{R} -subcoalgebra of C for every simple subcoalgebra D of C , C_0B is an \mathcal{R} -subcoalgebra by property (R2) where C_0 is the coradical of C . It follows from (R3) that $\bigwedge^n C_0B$ is an \mathcal{R} -subcoalgebra for any $n \geq 1$. Hence, $\sum_{n \geq 1} \bigwedge^n C_0B$ is an \mathcal{R} -subcoalgebra. The result follows from (R2) and the fact that

$$C = \sum_{n \geq 1} \bigwedge^n C_0B. \quad \square$$

In closing this section, we give several equivalent formulations of the idempotent radical class for the sake of convenience of further discussion.

Proposition 2.6. Let B be a bialgebra, and \mathcal{R} a non-empty collection of objects in $\mathbf{Coalg}\text{-}B$ which satisfies the conditions (R1) and (R3). Then \mathcal{R} is an idempotent radical class of $\mathbf{Coalg}\text{-}B$ if, and only if, one the following conditions holds:

- (R2) For any $C \in \mathbf{Coalg}\text{-}B$, if $\{C_i\}_{i \in I}$ is a family of \mathcal{R} -subcoalgebras of C , then $\sum_i C_i$ is an \mathcal{R} -subcoalgebra of C .
- (R2a) For any $C \in \mathbf{Coalg}\text{-}B$, if $\{C_i\}_{i \in I}$ is a chain of \mathcal{R} -subcoalgebras of C , then $\bigcup_i C_i$ is also an \mathcal{R} -subcoalgebra of C .
- (R2b) For any $C \in \mathbf{Coalg}\text{-}B$, there exists a maximal \mathcal{R} -subcoalgebra of C .

Proof. It suffices to show that (R2), (R2a) and (R2b) are equivalent conditions under the hypotheses (R1) and (R3). Obviously, (R2a) is an immediate consequence of (R2). By Zorn's Lemma, (R2b) follows from (R2a). Suppose that the condition (R2b) holds for \mathcal{R} . Let us have $C \in \mathbf{Coalg}\text{-}B$ and D a maximal \mathcal{R} -subcoalgebra of C . We claim that D contains all the \mathcal{R} -subcoalgebras of C . If not, there exists an \mathcal{R} -subcoalgebra D' of C such that $D' \not\subseteq D$. Then

$$D \subsetneq D' + D \subseteq D' \wedge D.$$

By (R3), $D' \wedge D$ is also an \mathcal{R} -subcoalgebra of C which properly contains D . This contradicts the maximality of D .

Let $\{C_i\}$ be a family of \mathcal{R} -subcoalgebras of C . Then $C_i \subseteq D$ for $i \in I$. Therefore, $\sum_{i \in I} C_i \subseteq D$. It follows from (R1) that $\sum_{i \in I} C_i$ is also an \mathcal{R} -subcoalgebra of C . Hence, (R2) holds for \mathcal{R} . \square

3. Projective radical

In this section, we will show that if H is a Hopf algebra, the collection \mathcal{P} of $(H-)$ projective H -module coalgebras is an idempotent radical class of $\mathbf{Coalg}\text{-}H$. We begin with a short review on Hopf modules.

Let B be a bialgebra and $C \in \mathbf{Coalg}\text{-}B$. Recall that a (right) (C, B) -Hopf module M is a right B -module and a right C -comodule such that the C -comodule structure map $\rho : M \rightarrow M \otimes C$ is a B -module map, where $M \otimes C$ uses the diagonal B -action. The category of (C, B) -Hopf modules is denoted by \mathcal{M}_B^C . The morphisms between two (C, B) -Hopf modules are those B -module maps which are also C -comodule maps at the same time. The category ${}^C\mathcal{M}_B$ can also be defined similarly, and both ${}^C\mathcal{M}_B$ and \mathcal{M}_B^C are abelian categories.

Recall the “wedge” product for comodules from [8]. Suppose $C \in \mathbf{Coalg}\text{-}B$ and $M \in \mathcal{M}_B^C$. For k -subspaces $N \subseteq M$ and $X \subseteq C$, the wedge product $N \wedge X$ is defined to be the kernel of the map

$$M \xrightarrow{\rho_M} M \otimes C \rightarrow M/N \otimes C/X.$$

If N is a B -submodule of M and X is a (C, B) -Hopf submodule of C , then $N \wedge X$ is also a (C, B) -Hopf submodule of M . If $M = C$ and both N and X are B -submodule coalgebras of C , then $N \wedge X$ is a B -submodule coalgebra of C .

Lemma 3.1. *Let B be a bialgebra and $C \in \mathbf{Coalg}\text{-}B$. Suppose that C_1, C_2 are B -submodule coalgebras of C . Then for any $M \in \mathcal{M}_B^{C_1 \wedge C_2}$, the sequence*

$$0 \rightarrow \{0\} \wedge C_1 \rightarrow M \rightarrow M/(\{0\} \wedge C_1) \rightarrow 0$$

is exact in $\mathcal{M}_B^{C_1 \wedge C_2}$. Moreover, $\{0\} \wedge C_1 \in \mathcal{M}_B^{C_1}$ and $M/(\{0\} \wedge C_1) \in \mathcal{M}_B^{C_2}$.

Proof. Notice that C_1, C_2 are B -submodule coalgebras of $C_1 \wedge C_2$. Hence, $\{0\} \wedge C_1$ is a $(C_1 \wedge C_2, B)$ -Hopf submodule of M . Since $\mathcal{M}_B^{C_1 \wedge C_2}$ is an abelian category, the quotient $M/(\{0\} \wedge C_1)$ is also a $(C_1 \wedge C_2, B)$ -Hopf module and the exact sequence follows easily. It follows immediately from the definition of wedge product that

$$\rho_M(\{0\} \wedge C_1) \subseteq (\{0\} \wedge C_1) \otimes C_1,$$

where $\rho_M : M \rightarrow M \otimes C$ is the $(C_1 \wedge C_2)$ -comodule structure map of M . Therefore, $\{0\} \wedge C_1 \in \mathcal{M}_B^{C_1}$. Since

$$M = \{0\} \wedge (C_1 \wedge C_2),$$

by the associativity of wedge product (cf. [8, Proposition 9.0.0]),

$$M = (\{0\} \wedge C_1) \wedge C_2.$$

Let $\eta_1 : M \rightarrow M/(\{0\} \wedge C_1)$ and $\eta_2 : C_1 \wedge C_2 \rightarrow (C_1 \wedge C_2)/C_2$ be the natural surjections. Then

$$(\eta_1 \otimes \eta_2) \circ \rho_M = 0.$$

Hence,

$$(\eta_1 \otimes 1) \circ \rho_M(M) \subseteq M/(\{0\} \wedge C_1) \otimes C_2.$$

Therefore, $M/(\{0\} \wedge C_1) \in \mathcal{M}_B^{C_2}$. \square

Theorem 3.2. *Let H be a Hopf algebra, and \mathcal{P} the collection of all right H -module coalgebras which are projective H -modules. Then \mathcal{P} is an idempotent radical class of $\mathbf{Coalg}\text{-}H$.*

Proof. Let $C \in \mathbf{Coalg}\text{-}H$. Recall from [1, Corollary 1] that C is a projective H -module if, and only if, there exists a right H -module map $\psi : C \rightarrow H$ such that

$$\varepsilon_H \circ \psi = \varepsilon_C. \quad (3.1)$$

In this case all the right (C, H) -Hopf modules are projective H -modules. By Proposition 2.6, it suffices to show that \mathcal{P} satisfies (R1), (R3) and (R2b).

(R1) Let C be a \mathcal{P} -coalgebra. For any H -submodule coalgebra D of C , D is a right (C, H) -Hopf module and hence a projective H -module.

(R3) Let $C \in \mathbf{Coalg}\text{-}H$, and let C_1, C_2 be \mathcal{P} -subcoalgebras of C . Then, by the preceding remark, $C_1 \wedge C_2$ is also a right H -module coalgebra. Now, by Lemma 3.1, the sequence

$$0 \longrightarrow C_1 \longrightarrow C_1 \wedge C_2 \longrightarrow (C_1 \wedge C_2)/C_1 \longrightarrow 0$$

is exact in $\mathcal{M}_H^{C_1 \wedge C_2}$. Moreover, $(C_1 \wedge C_2)/C_1 \in \mathcal{M}_H^{C_2}$. Therefore, $(C_1 \wedge C_2)/C_1$ is a projective H -module, and so the sequence is split exactly in \mathcal{Mod}_H . Thus $C_1 \wedge C_2 \cong C_1 \oplus (C_1 \wedge C_2)/C_1$ as H -modules. Hence $C_1 \wedge C_2$ is a projective H -module.

(R2b) Let us have $C \in \mathbf{Coalg}\text{-}H$ and S the set of all pairs (D, ψ) in which D is a \mathcal{P} -subcoalgebra of C and $\psi : D \longrightarrow H$ is a right H -module map satisfying (3.1). We define the partial ordering \leq on the non-empty set S as follows:

$$(D, \psi) \leq (D', \psi') \quad \text{if } D \subseteq D' \quad \text{and} \quad \psi'|_D = \psi.$$

Suppose that $\{(D_i, \psi_i)\}$ is a chain in S . Let $\overline{D} = \bigcup_i D_i$ and define $\overline{\psi} : \overline{D} \longrightarrow H$ as

$$\overline{\psi}(x) = \psi_i(x) \quad \text{if } x \in D_i.$$

Obviously, \overline{D} is a right H -submodule coalgebra of C . Since $\{(D_i, \psi_i)\}$ is a chain, the function $\overline{\psi} : \overline{D} \longrightarrow H$ is a well-defined right H -module map which satisfies (3.1), and so $(\overline{D}, \overline{\psi}) \in S$. By Zorn's Lemma, there is a maximal element $(D, \psi) \in S$. It follows from [1] that D is a \mathcal{P} -subcoalgebra of C . We now claim that D is a maximal \mathcal{P} -subcoalgebra of C . If the claim is false, then there exists a \mathcal{P} -subcoalgebra D' of C such that $D \subsetneq D'$. Since $D'/D \in \mathcal{M}_H^{D'}$, D'/D is a right projective H -module and so the sequence

$$0 \longrightarrow D \longrightarrow D' \longrightarrow D'/D \longrightarrow 0$$

is split exactly in \mathcal{Mod}_H . Therefore, $D' = D \oplus M$ for some H -submodule M of D' . Let $\psi' : D' \longrightarrow H$ be a right H -module map satisfying (3.1). Consider the map $\hat{\psi} = \psi \oplus \psi'|_M$. Then $\hat{\psi} : D' \longrightarrow H$ is also a right H -module map satisfying (3.1) and $\hat{\psi}|_D = \psi$. This leads to $(D, \psi) \not\leq (D', \psi)$, a contradiction! \square

The following corollary generalizes [5, Corollary 10] to arbitrary Hopf algebras.

Corollary 3.3. *Let H be a Hopf algebra. For every H -module coalgebra C , there exists a unique maximal projective H -submodule coalgebra $\mathcal{P}(C)$ of C . Moreover, the following statements about a right H -module coalgebra C are equivalent:*

- (i) C is H -projective;
- (ii) there exists an H -projective submodule coalgebra of C which contains the coradical C_0 ;
- (iii) DH is H -projective for every simple subcoalgebra D of C .

Proof. The corollary follows directly from Theorem 3.2 and Proposition 2.5. \square

As an application of the projective radical of H -module coalgebra, we give another proof for a projectivity result obtained by Schneider in [7, Theorem 4.11].

Corollary 3.4 (Schneider). *Let H be a Hopf algebra and C a right H -module coalgebra over the field k . Assume that:*

- (i) the coradical of $C \otimes k'$ is contained in $G(C \otimes k')(H \otimes k')$ for an extension $k' \supseteq k$, and
- (ii) the canonical map $\text{can} : C \otimes H \longrightarrow C \otimes C$ defined by

$$\text{can} : c \otimes b \mapsto \sum c_1 \otimes c_2 b$$

is injective.

Then C is a projective H -module.

Proof. Since C is H -projective if, and only if, $C \otimes k'$ is $H \otimes k'$ -projective for some extension $k' \supseteq k$, we may simply assume $k = k'$.

For $g \in G(C)$, the surjective map $\phi : H \longrightarrow gH$ defined by $\phi(h) = gh$ is an H -module coalgebra map. The injectivity of can implies that ϕ is also an isomorphism. In particular, gH is an H -free submodule coalgebra of C . By Theorem 3.2 and (R2), $G(C)H = \sum_{g \in G(C)} gH$ is an H -projective submodule coalgebra of C , and it contains the coradical of C . It follows from Corollary 3.3 that C is a projective H -module. \square

4. \mathcal{E} is an idempotent radical class

Let C be a B -module coalgebra. Then $\bar{C} = C/CB^+$ admits a natural coalgebra structure, where $B^+ = \ker \varepsilon_B$, and the natural surjection $\eta_C : C \rightarrow \bar{C}$ is a coalgebra map. For any right B -module M , we let

$$\bar{M} = M/(MB^+),$$

and $\eta_M : M \rightarrow \bar{M}$, $m \mapsto \bar{m}$, the quotient map. If $M \in \mathcal{M}_B^C$, then \bar{M} admits a natural right \bar{C} -comodule structure, and $\bar{?} : \mathcal{M}_B^C \rightarrow \mathcal{M}^{\bar{C}}$, $M \mapsto \bar{M}$, defines a k -linear functor.

For any $N \in \mathcal{M}^{\bar{C}}$, the cotensor product $N \square_{\bar{C}} C$ is defined as the kernel of the map

$$\rho_N \otimes \text{id}_C - \text{id}_N \otimes ((\eta_C \otimes \text{id}) \circ \Delta_C) : N \otimes C \rightarrow N \otimes \bar{C} \otimes C.$$

The cotensor product $N \square_{\bar{C}} C$ has a right (C, B) -Hopf module structure inherited from C , and $? \square_{\bar{C}} C : \mathcal{M}^{\bar{C}} \rightarrow \mathcal{M}_B^C$ is a k -linear functor. Moreover, the functor $\bar{?}$ is left adjoint to $? \square_{\bar{C}} C$ with the unit Ξ and counit Θ of the adjunction given by

$$\begin{aligned} \Xi_M : M &\rightarrow \bar{M} \square_{\bar{C}} C, m \mapsto \sum \bar{m}_0 \otimes m_1 \\ \Theta_N : N \square_{\bar{C}} C &\rightarrow N, \sum n_i \otimes c_i \mapsto \sum n_i \varepsilon(c_i) \end{aligned}$$

where $M \in \mathcal{M}_B^C$ and $N \in \mathcal{M}^{\bar{C}}$. In particular, $? \square_{\bar{C}} C$ is left exact and $\bar{?}$ is right exact (cf. [9, Theorem 2.6.1]).

In this section, we prove in Corollary 4.3 that $\bar{?} : \mathcal{M}_B^C \rightarrow \mathcal{M}^{\bar{C}}$ is an equivalence if, and only if, for all $M \in \mathcal{M}_B^C$, (i) $\text{Tor}_1^B(M, k) = 0$, and (ii) $M = 0$ whenever $\bar{M} = 0$. Using this characterization for the equivalence of $\bar{?}$, we show in Theorem 4.4 that the collection

$$\mathcal{E} = \{C \in \mathbf{Coalg}\text{-}B \mid \text{the functor } \bar{?} : \mathcal{M}_B^C \rightarrow \mathcal{M}^{\bar{C}} \text{ is an equivalence}\}$$

is an idempotent radical class of $\mathbf{Coalg}\text{-}B$. The result will be used to show that the B -cocleft module coalgebras also form an idempotent radical class. We begin with a characterization for the equivalence of the functor $\bar{?}$.

Proposition 4.1. *Let B be a bialgebra, and C a right B -module coalgebra. Then the following statements are equivalent:*

- (i) *the functor $\bar{?} : \mathcal{M}_B^C \rightarrow \mathcal{M}^{\bar{C}}$ is an equivalence;*
- (ii) *the functor $\bar{?}$ is exact, and for any $M \in \mathcal{M}_B^C$, $\bar{M} = 0$ implies $M = 0$;*
- (iii) *the unit Ξ and counit Θ of the adjunction are isomorphisms.*

Proof. The implications (i) \Rightarrow (ii) and (iii) \Rightarrow (i) are straightforward. It remains to show that (ii) implies (iii).

For any k -space V , $V \otimes C$ admits a right (C, B) -Hopf module structure induced by the B -action and comultiplication of C , and $\mathcal{F}(V) := V \otimes \bar{C}$ is a right \bar{C} -comodule with its structure inherited from the comultiplication of \bar{C} . The map

$$\mathcal{F}(V) \square_{\bar{C}} C \xrightarrow{\text{id}_V \otimes \varepsilon \otimes \text{id}_C} V \otimes C$$

is an isomorphism of (C, B) -Hopf modules. Note that $\tilde{\eta} : \bar{V} \otimes \bar{C} \rightarrow \mathcal{F}(V)$, defined by $\tilde{\eta} : \bar{v} \otimes \bar{c} \mapsto v \otimes \bar{c}$, is an isomorphism of right \bar{C} -comodules, and the diagram

$$\begin{array}{ccc} \mathcal{F}(V) \square_{\bar{C}} C & \xrightarrow{\text{id}_V \otimes \varepsilon \otimes \text{id}_C} & \bar{V} \otimes \bar{C} \\ & \searrow \Theta_{\mathcal{F}(V)} \quad \swarrow \tilde{\eta} & \\ & \mathcal{F}(V) & \end{array}$$

commutes. Therefore, $\Theta_{\mathcal{F}(V)}$ is an isomorphism.

For $N \in \mathcal{M}^{\bar{C}}$, the \bar{C} -comodule structure map $\rho_N : N \rightarrow \mathcal{F}(N)$ is an injective \bar{C} -comodule map. Let

$$N' := \text{coker } \rho_N = \mathcal{F}(N)/\rho_N(N),$$

and let g be the composition

$$\mathcal{F}(N) \xrightarrow{\pi} N' \xrightarrow{\rho_{N'}} \mathcal{F}(N'),$$

where π is the natural surjection map. Then we have the exact sequence

$$0 \rightarrow N \xrightarrow{\rho_N} \mathcal{F}(N) \xrightarrow{g} \mathcal{F}(N')$$

in $\mathcal{M}^{\bar{C}}$.

Assume (ii) holds. Since $?\square_{\bar{C}}C$ is left exact and $\bar{?}$ is exact, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overline{N \square_{\bar{C}} C} & \longrightarrow & \overline{\mathcal{F}(N) \square_{\bar{C}} C} & \longrightarrow & \overline{\mathcal{F}(N') \square_{\bar{C}} C} \\ & & \downarrow \Theta_N & & \downarrow \Theta_{\mathcal{F}(N)} & & \downarrow \Theta_{\mathcal{F}(N')} \\ 0 & \longrightarrow & N & \longrightarrow & \mathcal{F}(N) & \longrightarrow & \mathcal{F}(N') \end{array}$$

where the rows are exact, and both $\Theta_{\mathcal{F}(N)}$ and $\Theta_{\mathcal{F}(N')}$ are isomorphisms. It follows from diagram tracing that Θ_N is also an isomorphism.

For any $M \in \mathcal{M}_B^C$, we have the exact sequence

$$0 \longrightarrow \ker \Xi_M \longrightarrow M \xrightarrow{\Xi_M} \overline{M \square_{\bar{C}} C} \longrightarrow \operatorname{coker} \Xi_M \longrightarrow 0$$

in \mathcal{M}_B^C . By the exactness of $\bar{?}$, we also have the exact sequence

$$0 \longrightarrow \overline{\ker \Xi_M} \longrightarrow \overline{M} \xrightarrow{\bar{\Xi}_M} \overline{M \square_{\bar{C}} C} \longrightarrow \overline{\operatorname{coker} \Xi_M} \longrightarrow 0$$

in $\mathcal{M}^{\bar{C}}$. Since the equation

$$\Theta_{\overline{M}} \circ \bar{\Xi}_M = \operatorname{id}_{\overline{M}}$$

holds for every $M \in \mathcal{M}_B^C$ and $\Theta_{\overline{M}}$ is an isomorphism, so is $\bar{\Xi}_M$. Therefore,

$$\overline{\ker \Xi_M} = 0 = \overline{\operatorname{coker} \Xi_M}.$$

It follows from (ii) that

$$\ker \Xi_M = 0 = \operatorname{coker} \Xi_M.$$

Hence, Ξ_M is an isomorphism. \square

Proposition 4.2. Let B be a bialgebra and C a right B -module coalgebra. Then the functor $\bar{?} : \mathcal{M}_B^C \longrightarrow \mathcal{M}^{\bar{C}}$ is exact if, and only if, $\operatorname{Tor}_1^B(M, k) = 0$ for all $M \in \mathcal{M}_B^C$, where k is considered as a trivial left B -module.

Proof. Let

$$0 \rightarrow M_1 \xrightarrow{a} M_2 \xrightarrow{b} M_3 \rightarrow 0$$

be an exact sequence in \mathcal{M}_B^C . By the associated long exact sequence, the sequence

$$\operatorname{Tor}_1^B(M_3, k) \rightarrow M_1 \otimes_B k \xrightarrow{a \otimes_B k} M_2 \otimes_B k \xrightarrow{b \otimes_B k} M_3 \otimes_B k \rightarrow 0 \quad (4.1)$$

is exact. Note that $\phi_M : \overline{M} \longrightarrow M \otimes_B k$, defined by $\phi : \bar{m} \mapsto m \otimes 1$ for $m \in M$, is a natural isomorphism of k -linear spaces. If $\operatorname{Tor}_1^B(M, k) = 0$ for all $M \in \mathcal{M}_B^C$, then it follows from (4.1) that the sequence

$$0 \rightarrow M_1 \otimes_B k \xrightarrow{a \otimes_B k} M_2 \otimes_B k \xrightarrow{b \otimes_B k} M_3 \otimes_B k \rightarrow 0$$

is exact. By the naturality of ϕ , the sequence

$$0 \rightarrow \overline{M_1} \xrightarrow{\bar{a}} \overline{M_2} \xrightarrow{\bar{b}} \overline{M_3} \rightarrow 0$$

is exact in $\mathcal{M}^{\bar{C}}$. Therefore, $\bar{?}$ is an exact functor.

Conversely, assume that $\bar{?}$ is exact. For $M \in \mathcal{M}_B^C$ with C -comodule structure $\rho_M : M \rightarrow M \otimes C$, let $\mathcal{G}(M) := M \otimes B$ be the right (C, B) -Hopf module with the B -action \cdot , and the C -coaction $\rho_{\mathcal{G}(M)} : \mathcal{G}(M) \rightarrow \mathcal{G}(M) \otimes C$ given by

$$(m \otimes b) \cdot h = m \otimes bh \quad \text{and} \\ \rho_{\mathcal{G}(M)}(m \otimes b) = \sum m_0 \otimes b_1 \otimes m_1 b_2$$

for all $m \in M$ and $b, h \in B$, where $\rho_M(m) = \sum m_0 \otimes m_1$ and $\Delta(b) = \sum b_1 \otimes b_2$. Notice that the B -module structure map $\mu : \mathcal{G}(M) \rightarrow M$ of M is a (C, B) -Hopf module map, and so we have the exact sequence

$$0 \longrightarrow \ker \mu \xrightarrow{i} \mathcal{G}(M) \xrightarrow{\mu} M \longrightarrow 0$$

in \mathcal{M}_B^C , where i is the inclusion map. By the naturality of ϕ , we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \mu & \xrightarrow{\bar{i}} & \mathcal{G}(M) & \xrightarrow{\bar{\mu}} & M \\ & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ \text{Tor}_1^B(\mathcal{G}(M), k) & \longrightarrow & \text{Tor}_1^B(M, k) & \longrightarrow & \ker \mu \otimes_B k & \xrightarrow{i \otimes_B k} & \mathcal{G}(M) \otimes_B k \xrightarrow{\mu \otimes_B k} M \otimes_B k, \end{array} \quad (4.2)$$

where the top and bottom rows are exact by the exactness of $\bar{?}$ and the associated long exact sequence respectively. In particular, $\ker(i \otimes_B k) = 0$ and so the map

$$\text{Tor}_1^B(\mathcal{G}(M), k) \longrightarrow \text{Tor}_1^B(M, k)$$

in the diagram is surjective. Since $\mathcal{G}(M)$ is a free right B -module, $\text{Tor}_1^B(\mathcal{G}(M), k) = 0$. Therefore, $\text{Tor}_1^B(M, k) = 0$. \square

Corollary 4.3. *Let B be a bialgebra, and C a B -module coalgebra. Then the functor $\bar{?} : \mathcal{M}_B^C \rightarrow \mathcal{M}^{\bar{C}}$ is an equivalence if, and only if, the following conditions hold for all $M \in \mathcal{M}_B^C$:*

- (i) $\text{Tor}_1^B(M, k) = 0$, and
- (ii) $\bar{M} = 0$ implies $M = 0$.

Proof. The statement follows immediately from Propositions 4.1 and 4.2. \square

Now we turn to the main result of this section.

Theorem 4.4. *Let B be a bialgebra and \mathcal{E} the collection of all the B -module coalgebras C such that $\bar{?} : \mathcal{M}_B^C \rightarrow \mathcal{M}^{\bar{C}}$ is an equivalence. Then \mathcal{E} is an idempotent radical class of **Coalg- B** .*

Proof. By Proposition 2.6, it suffices to show that \mathcal{E} satisfies (R1), (R2a) and (R3).

(R1) Let C be an \mathcal{E} -coalgebra, and D a B -submodule coalgebra of C . For $M \in \mathcal{M}_B^D$, $M \in \mathcal{M}_B^C$. By Corollary 4.3, $\bar{?} : \mathcal{M}_B^D \rightarrow \mathcal{M}^{\bar{D}}$ is an equivalence.

(R2a) Let C be a right B -module coalgebra and $\{C_i\}$ a chain of \mathcal{E} -subcoalgebras of C . Then $D := \bigcup_i C_i$ is a B -submodule coalgebra of C . For $M \in \mathcal{M}_B^D$, let $M_i = \{0\} \wedge C_i$. Then $M_i \in \mathcal{M}_B^{C_i}$ and

$$M = \bigcup_i M_i. \quad (4.3)$$

It follows from Corollary 4.3 that

$$\text{Tor}_1^B(M_i, k) = 0$$

for all i . Since $\text{Tor}_1^B(-, k)$ commutes with direct limit (cf. [9, Corollary 2.6.17]), we have

$$\text{Tor}_1^B(M, k) = \lim_{\rightarrow} \text{Tor}_1^B(M_i, k) = 0.$$

Therefore, the functor $\bar{?} : \mathcal{M}_B^D \longrightarrow \mathcal{M}^{\bar{D}}$ is exact by Proposition 4.2. Suppose $\bar{M} = 0$. By the exactness of $\bar{?}$, the sequence

$$0 \longrightarrow \overline{M_i} \xrightarrow{\bar{j}} \bar{M}$$

is exact for all i , where j is the inclusion map. Hence, $\overline{M_i} = 0$ for all i . By Corollary 4.3, $M_i = 0$ for all i . It follows from (4.3) that $M = 0$. Therefore, by Corollary 4.3, D is an \mathcal{E} -subcoalgebra of C .

(R3) Let C_1, C_2 be the \mathcal{E} -subcoalgebra of a B -module coalgebra C , and $D' = C_1 \wedge C_2$. For $M \in \mathcal{M}_B^{D'}$, by Lemma 3.1, we have the exact sequence

$$0 \longrightarrow \{0\} \wedge C_1 \longrightarrow M \longrightarrow M/(\{0\} \wedge C_1) \longrightarrow 0 \quad (4.4)$$

in $\mathcal{M}_B^{D'}$ with $\{0\} \wedge C_1 \in \mathcal{M}_B^{C_1}$ and $M/(\{0\} \wedge C_1) \in \mathcal{M}_B^{C_2}$. In particular,

$$\mathrm{Tor}_1^B(M/(\{0\} \wedge C_1), k) = 0 = \mathrm{Tor}_1^B(\{0\} \wedge C_1, k).$$

By the associated long exact sequence of (4.4), we have the exact sequence

$$\mathrm{Tor}_1^B(\{0\} \wedge C_1, k) \longrightarrow \mathrm{Tor}_1^B(M, k) \longrightarrow \mathrm{Tor}_1^B(M/(\{0\} \wedge C_1), k).$$

Therefore, $\mathrm{Tor}_1^B(M, k) = 0$. Hence by Proposition 4.2, the functor $\bar{?} : \mathcal{M}_B^{D'} \longrightarrow \mathcal{M}^{\bar{D}'}$ is exact. Suppose $\bar{M} = 0$. Then, by (4.4) and the exactness of $\bar{?}$, we have

$$\overline{\{0\} \wedge C_1} = 0 = \overline{M/(\{0\} \wedge C_1)}.$$

Since C_1, C_2 are \mathcal{E} -coalgebras, it follows from Corollary 4.3 that

$$\{0\} \wedge C_1 = 0 = M/(\{0\} \wedge C_1)$$

and hence $M = 0$. By Corollary 4.3 again, D' is an \mathcal{E} -coalgebra. \square

5. Cocleft radical

In this section, we prove that the collection \mathcal{C} of all cocleft B -module coalgebras is an idempotent radical class. As an application, we use this result to give another proof for a normal basis theorem of Schneider [7, Theorem III] without assuming a bijective antipode of the underlying Hopf algebra. We begin with some definitions and properties of cocleft B -module coalgebras.

Definition 5.1. Let B be a bialgebra and C a right B -module coalgebra.

(i) C is said to have a *normal basis* if we have

$$C \cong \bar{C} \otimes B \quad \text{as left } \bar{C}\text{-comodules and right } B\text{-modules.}$$

(ii) C is said to be *B -coGalois* if the map

$$can : C \otimes B \longrightarrow C \square_{\bar{C}} C, \quad c \otimes b \mapsto \sum c_1 \otimes c_2 b$$

is a k -linear isomorphism.

(iii) A B -module map $\gamma : C \longrightarrow B$ is called a *cointegral* if γ is convolution invertible.

(iv) C is said to be *B -cocleft* if it admits a cointegral.

(v) A B -submodule coalgebra D of C is called a *B -cocleft subcoalgebra* if D is B -cocleft.

It was proved in [3, Theorem 2.3] that C is B -cocleft if, and only if, C is B -coGalois and has a normal basis. In this case, $\bar{?} : \mathcal{M}_B^C \longrightarrow \mathcal{M}^{\bar{C}}$ is an equivalence and all the (C, B) -Hopf modules are free B -modules. In particular, a B -cocleft coalgebra is an \mathcal{E} -coalgebra or $\mathcal{C} \subset \mathcal{E}$.

Lemma 5.2. Let B be a bialgebra, and C an \mathcal{E} -coalgebra. Suppose that for every simple subcoalgebra E of \bar{C} there exists a B -cocleft subcoalgebra D of C such that $E \subseteq \eta_C(D)$, where $\eta_C : C \longrightarrow \bar{C}$ is the natural surjection. Then C is B -cocleft.

Proof. By Proposition 4.1, both Θ and Ξ are isomorphisms and so $\square_{\bar{C}}C : \mathcal{M}_{\bar{C}}^{\bar{C}} \rightarrow \mathcal{M}_B^C$ is a k -linear equivalence. In particular, C is an injective left \bar{C} -comodule. In order to show that C is B -cocleft, by [3, Theorem 2.3], it suffices to show that C admits a normal basis. By [7, Corollary 2.2], it is enough to show that $E \square_{\bar{C}} C \cong E \otimes B$ as left E -comodules and right B -modules for all simple subcoalgebras E of \bar{C} .

Let E be a simple subcoalgebra of \bar{C} , and D a B -cocleft subcoalgebra of C such that $E \subseteq \eta_C(D)$. Let $i : D \rightarrow C$ be the inclusion map. Then i is a monomorphism in \mathcal{M}_B^C and $\bar{i}(\bar{D}) = \eta_C(D)$. Since $\bar{\gamma} : \mathcal{M}_B^C \rightarrow \mathcal{M}_{\bar{C}}^{\bar{C}}$ is exact, $\bar{i} : \bar{D} \rightarrow \eta_C(D)$ is a coalgebra isomorphism. Thus E can be viewed as a right \bar{D} -comodule via \bar{i} . Notice that $E \square_{\bar{D}} D$ is a (C, B) -Hopf submodule of $E \square_{\bar{C}} C$. Let $j : E \square_{\bar{D}} D \rightarrow E \square_{\bar{C}} C$ be the inclusion map. Then we have the following commutative diagram:

$$\begin{array}{ccc} E \square_{\bar{D}} D & \xrightarrow{j} & E \square_{\bar{C}} C \\ \Theta_E \downarrow & & \downarrow \Theta_E \\ E & \xlongequal{\quad} & E \end{array}$$

of k -linear maps. Since both D and C are \mathcal{E} -coalgebra, the vertical maps Θ_E in the diagram are isomorphisms by Proposition 4.1. Hence, \bar{j} is an isomorphism. Since $\bar{\gamma}$ is an equivalence, j is also an isomorphism and so

$$E \square_{\bar{C}} C = E \square_{\bar{D}} D.$$

Since D has a normal basis, it follows from [7, Corollary 2.2] that

$$E \square_{\bar{C}} C = E \square_{\bar{D}} D \cong E \otimes B$$

as left E -comodules and right B -modules. \square

Now we can prove our main theorem of this section.

Theorem 5.3. *Let B be a bialgebra. Then the collection \mathcal{C} of all B -cocleft coalgebras is an idempotent radical class of $\mathbf{Coalg}\text{-}B$.*

Proof. By Proposition 2.6, we only need to show that \mathcal{C} satisfies (R1), (R2a) and (R3).

(R1) Let C be a B -cocleft coalgebra. Then there exists a cointegral $\gamma : C \rightarrow B$. For any non-zero B -submodule coalgebra D of C , $\gamma|_D : D \rightarrow B$ is clearly a cointegral of D . Hence D is also B -cocleft.

(R2a) Let C be a B -module coalgebra and $\{C_i\}$ a chain of B -cocleft subcoalgebras of C . Since C_i is an \mathcal{E} -coalgebra for all i , by Theorem 4.4 and (R2), $D = \bigcup_i C_i = \sum_i C_i$ is also an \mathcal{E} -coalgebra. Note that $\bar{D} = \bigcup_i \eta_D(C_i)$. Since every simple subcoalgebra E of \bar{D} is finite dimensional, E is contained in $\eta_D(C_i)$ for some i . By Lemma 5.2, D is B -cocleft.

(R3) Let C_1, C_2 be B -cocleft subcoalgebras of a right B -module coalgebra C and $D' = C_1 \wedge C_2$. By Theorem 4.4, D' is an \mathcal{E} -coalgebra. By [8, Lemma 9.1.3],

$$\bar{D}' = \eta_{D'}(D') \subseteq \eta_{D'}(C_1) \wedge \eta_{D'}(C_2) \subseteq \bar{D}'.$$

For any simple coalgebra E of \bar{D}' , $E \subseteq \eta_{D'}(C_1)$ or $E \subseteq \eta_{D'}(C_2)$. Hence, by Lemma 5.2, D' is B -cocleft. \square

Corollary 5.4. *Let B be a bialgebra. Then every right B -module coalgebra C admits a unique maximal B -cocleft subcoalgebra $\mathcal{C}(C)$ of C , and $\mathcal{C}(C) = \mathcal{C}(C) \wedge \mathcal{C}(C)$. Moreover, the following statements about a right B -module coalgebra C are equivalent:*

- (a) C is B -cocleft;
- (b) there exists a B -cocleft subcoalgebra of C which contains the coradical of C ;
- (c) EB is B -cocleft for every simple subcoalgebra E of C .

Proof. By Theorem 5.3, the collection of all B -cocleft coalgebras is an idempotent radical class of $\mathbf{Coalg}\text{-}B$. The corollary follows immediately from Proposition 2.5. \square

In closing of this paper, we give another proof for a normal basis theorem of Schneider [7, Theorem III] without assuming a bijective antipode.

Corollary 5.5. *Let H be a Hopf algebra and C a right H -module coalgebra. Suppose that $\text{can} : C \otimes H \longrightarrow C \square_C C$ is injective and $G(C)H$ contains the coradical of C . Then C is H -cocleft.*

Proof. For $g \in G(C)$, the injectivity of can implies that $\phi : H \longrightarrow gH$, $h \mapsto gh$ is an H -module isomorphism. Let $\psi : gH \longrightarrow H$ be the inverse of ϕ , and $\bar{\psi} = S \circ \psi$, where S is the antipode of H . Then

$$\psi * \bar{\psi}(gh) = \sum h_1 S(h_2) = \varepsilon(h) 1_H = \sum S(h_1) h_2 = \bar{\psi} * \psi(gh)$$

for $h \in H$. Therefore, ψ is a cointegral of gH . In particular, gH is H -cocleft. By Theorem 5.3 and (R2), $G(C)H = \sum_{g \in G(C)} gH$ is H -cocleft. Since $G(C)H$ contains the coradical of C , it follows from Corollary 5.4 that C is H -cocleft. \square

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